

# AN UPPER BOUND FOR NONNEGATIVE RANK

YAROSLAV SHITOV

ABSTRACT. We provide a nontrivial upper bound for the nonnegative rank of rank-three matrices, which allows us to prove that  $\lceil \frac{6n}{7} \rceil$  linear inequalities suffice to describe a convex  $n$ -gon up to a linear projection.

## 1. PRELIMINARIES

Consider a convex polytope  $P \subset \mathbb{R}^n$ . An *extension* [5, 8] of  $P$  is a polytope  $Q \subset \mathbb{R}^d$  such that  $P$  can be obtained from  $Q$  as an image under a linear projection from  $\mathbb{R}^d$  to  $\mathbb{R}^n$ . An *extended formulation* [8, 10] of  $P$  is a description of  $Q$  by linear equations and linear inequalities (together with the projection). The *size* [8, 10] of the extended formulation is the number of facets of  $Q$ . The *extension complexity* [8, 10] of a polytope  $P$  is the smallest size of any extended formulation of  $P$ , that is, the minimal possible number of inequalities in the description of  $Q$ . The number of facets of  $Q$  can sometimes be significantly smaller [5] than that of  $P$ , and this phenomenon can be used to reduce the complexity of linear programming problems useful for numerous applications [3, 5, 10].

An important result providing the linear algebraic characterization of extended formulations has been obtained in 1991 by Yannakakis [12]. Let a polytope  $P$  (with  $v$  vertices and  $f$  facets) be defined as the set of all points  $x \in \mathbb{R}^n$  satisfying the conditions  $c_i(x) \geq \beta_i$  and  $c_j(x) = \beta_j$ , for  $i \in \{1, \dots, f\}$  and  $j \in \{f+1, \dots, q\}$ , where  $c_1, \dots, c_q$  are linear functionals on  $\mathbb{R}^n$ . A slack matrix  $S = S(P)$  of  $P$  is an  $f$ -by- $v$  matrix satisfying  $S_{it} = c_i(p_t) - \beta_i$ , where  $p_1, \dots, p_v$  denote the vertices of  $P$ , and we note that  $S$  is nonnegative. The following well-known result (see [8, Corollary 5] and also [7, Lemma 3.1]) characterizes the rank of  $S(P)$  in terms of the dimension of  $P$ .

**Proposition 1.1.** *A slack matrix of a polytope  $P$  has classical rank one greater than the dimension of  $P$ .*

The result by Yannakakis points out the connection between extension complexity and nonnegative factorizations and can now be formulated as follows [8, 10, 12].

**Theorem 1.2.** [10, Theorem 2] *The extension complexity of a polytope  $P$  is equal to the minimal  $k$  for which  $S(P)$  can be written as a product of  $f$ -by- $k$  and  $k$ -by- $v$  nonnegative matrices.*

In general, the smallest integer  $k$  for which there exists a factorization  $A = BC$  with  $B \in \mathbb{R}_+^{n \times k}$  and  $C \in \mathbb{R}_+^{k \times m}$  is called the *nonnegative rank* of a nonnegative matrix  $A \in \mathbb{R}_+^{n \times m}$ . Nonnegative factorizations are being widely studied and used in data analysis, statistics, computational biology, clustering and numerous other applications [2]. There are still many open questions on nonnegative rank interesting for different applications, and a considerable part of them is related to providing the bounds on the nonnegative rank in terms of other matrix invariants [4, 8, 10].

In fact, it has still been unknown whether any nontrivial upper bound for the nonnegative rank exists in terms of the classical rank function. It is easy to show that the nonnegative rank of a matrix equals [2] the classical rank if one of them is less than 3. However, even for a rank-three  $m$ -by- $n$  matrix, no upper bound (instead of  $\min\{m, n\}$ , which is trivial) for the nonnegative rank has been known.

**Problem 1.3.** [1, Conjecture 3.2] Assume  $n \geq 3$ . Does there exist a rank-three  $n$ -by- $n$  nonnegative matrix with nonnegative rank equal to  $n$ ?

In view of Proposition 1.1 and Theorem 1.2, one can ask a related question on whether there exists a convex  $n$ -gon with extension complexity equal to  $n$ , for every  $n$ . For  $n \leq 5$ , Problem 1.3 has been solved in the positive in [8]. In [6] it was noted that a sufficiently irregular convex hexagon has full extension complexity, stating the positive answer for  $n = 6$ . For  $n \geq 7$ , the problem has been open.

Lin and Chu [11] claimed a positive answer for Problem 1.3, but their argument has been shown to contain a gap [8, 9]. A negative answer for Problem 1.3 has been obtained in [8] for a special case of so-called Euclidean distance matrices. The factorizations of those matrices have been studied subsequently in [9], and the logarithmic upper bounds have been obtained in a number of important special cases. A detailed investigation of extended formulations of convex polygons has been undertaken in [5], but the question about an  $n$ -gon with extension complexity equal to  $n$  has also been left open.

In our paper we solve Problem 1.3 and prove that for  $n > 6$ , the answer is negative. In fact, we provide a nontrivial upper bound for the nonnegative rank of matrices in terms of classical rank and prove that an  $m$ -by- $n$  rank-three matrix cannot have nonnegative rank greater than  $\left\lceil \frac{6 \min\{m, n\}}{7} \right\rceil$ . We also answer the question on extension complexity and show that a convex  $n$ -gon has extension complexity at most  $\left\lceil \frac{6n}{7} \right\rceil$ . That is, we prove that any convex  $n$ -gon admits a description with  $\left\lceil \frac{6n}{7} \right\rceil$  linear inequalities up to a projection.

The organization of the paper is as follows. In the second section, we prove the main result in a special case of slack matrices of convex heptagons, thus showing that any convex heptagon admits a description with six linear inequalities. In the third section, we use those results and prove the main results of our paper, which include the upper bounds for the extension complexity of a polygon and for the nonnegative rank of a rank-three matrix.

## 2. FACTORING A SLACK MATRIX OF A CONVEX HEPTAGON

In this section, we will prove that slack matrices of convex heptagons have non-negative ranks less than 7. The considerations of this section deal with matrices having not more than seven rows and seven columns, and we adopt the following convention in order to make the presentation more concise.

**Convention 2.1.** Throughout this section, the row and column indexes of the matrices considered are to be understood as the elements of the ring  $\mathbb{Z}/7\mathbb{Z}$ . In particular,  $A_{3+6, 6+1}$  will stand for the  $(2, 7)$ th entry of a matrix  $A$ . Also, we will use the letters  $i$  and  $j$  only for denoting such indexes in the present section, and we operate with  $i$  and  $j$  as with elements from  $\mathbb{Z}/7\mathbb{Z}$ , throughout the section.

Let us introduce a certain special form of matrices which will be important for the considerations of the present section. By  $W[i, j, k]$  we denote the submatrix of  $W$  formed by the rows with indexes  $i, j$ , and  $k$ .

**Notation 2.2.** Given a real vector  $\alpha = (a_1, a_2, a_3, b_1, b_2, b_3)$ . By  $W(\alpha)$  we will denote the 7-by-3 matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & a_1 & a_2 & a_3 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}^\top,$$

and by  $\mathcal{V}(\alpha)$  the 7-by-7 matrix with  $(i, j)$ th entry equal to  $\det W[i-1, j-2, j-1]$ .

The following lemma points out a symmetry in the construction of  $\mathcal{V}$ .

**Lemma 2.3.** *Matrices  $\mathcal{V}(a_1, a_2, a_3, b_1, b_2, b_3)$  and  $\mathcal{V}(b_3, b_2, b_1, a_3, a_2, a_1)$  coincide up to relabeling the rows and columns.*

*Proof.* Perform the permutation (16)(25)(34) on the row indexes and (17)(26)(35) on the column indexes of  $\mathcal{V}(b_3, b_2, b_1, a_3, a_2, a_1)$ .  $\square$

Let us present a useful special case when the nonnegative rank of  $\mathcal{V}$  is not full.

**Lemma 2.4.** *Given a real vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j-1, j\}$ . If  $a_1 + b_1 \geq a_2 + b_2$  and  $a_3 + b_3 \geq a_2 + b_2$ , then  $V$  has nonnegative rank less than 7.*

*Proof.* One can check that  $V = FG$ , where

$$F = \begin{pmatrix} 0 & 0 & 1 & V_{41} + V_{47} & V_{61} & 0 \\ 0 & 0 & 0 & 1 & a_1 - a_2 + b_1 - b_2 & 1 \\ V_{31} & 0 & 0 & 1 & V_{37} & 0 \\ V_{41} & 1 & 0 & 0 & V_{47} & 0 \\ -a_2 + a_3 - b_2 + b_3 & 1 & 0 & 0 & 0 & 1 \\ V_{61} & V_{31} + V_{37} & 1 & 0 & 0 & 0 \\ 0 & V_{31} & 1 & V_{47} & 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & V_{32}/V_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{21}/V_{31} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{13} & 1 & V_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & V_{57}/V_{47} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{65}/V_{47} & 1 \\ V_{72} & 0 & 0 & 1 & 0 & 0 & V_{57} \end{pmatrix}.$$

$\square$

Now we show how can one construct new full-rank matrices from given.

**Lemma 2.5.** *Given a real vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j-1, j\}$ . Take  $\alpha_1 = (1 - a_3 - b_3)/(1 - b_3)$ ,  $\alpha_2 = (a_1 - a_1b_3 - a_3 + a_3b_1)/(a_1 - a_1b_3)$ ,  $\alpha_3 = (a_2 - a_2b_3 - a_3 + a_3b_2)/(a_2 - a_2b_3)$ ,  $\beta_1 = a_3$ ,  $\beta_2 = a_3/a_1$ ,  $\beta_3 = a_3/a_2$ . Then the matrix  $U = \mathcal{V}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$  satisfies  $U_{ij} > 0$  if  $i \notin \{j, j+1\}$  and has nonnegative rank equal to that of  $V$ .*

*Proof.* One can check that  $V = Q_1 U Q_2$ , where

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(1-b_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1/a_3 \\ a_2/a_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_1 a_2 (1-b_3)}{a_3} \\ a_2(1-b_3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(1-b_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-b_3}{a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{a_1(1-b_3)}{a_3} & 0 \end{pmatrix}.$$

Since the numbers  $1-b_3 = V_{42}$ ,  $a_1 = V_{63}$ ,  $a_2 = V_{73}$ , and  $a_3 = V_{13}$  are positive, the result follows.  $\square$

The following six real sequences will be important in our considerations.

**Notation 2.6.** Given a real vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j-1, j\}$ . We will consider the six sequences  $\alpha_1(t)$ ,  $\alpha_2(t)$ ,  $\alpha_3(t)$ ,  $\beta_1(t)$ ,  $\beta_2(t)$ , and  $\beta_3(t)$  of reals defined by  $\alpha_1(0) = a_1$ ,  $\alpha_2(0) = a_2$ ,  $\alpha_3(0) = a_3$ ,  $\beta_1(0) = b_1$ ,  $\beta_2(0) = b_2$ ,  $\beta_3(0) = b_3$ , and also

$$\alpha_1(t+1) = \frac{1 - \alpha_3(t) - \beta_3(t)}{1 - \beta_3(t)},$$

$$\alpha_{\chi+1}(t+1) = \frac{\alpha_\chi(t) - \alpha_\chi(t)\beta_3(t) - \alpha_3(t) + \alpha_3(t)\beta_\chi(t)}{\alpha_\chi(t) - \alpha_\chi(t)\beta_3(t)} \text{ for } \chi \in \{1, 2\},$$

$$\beta_1(t+1) = \alpha_3(t), \quad \beta_2(t+1) = \alpha_3(t)/\alpha_1(t), \quad \beta_3(t+1) = \alpha_3(t)/\alpha_2(t).$$

*Remark 2.7.* Lemma 2.5 shows that the sequences  $\alpha_1(t)$ ,  $\alpha_2(t)$ ,  $\alpha_3(t)$ ,  $\beta_1(t)$ ,  $\beta_2(t)$ , and  $\beta_3(t)$  are well defined.

It turns out that the sequences introduced are in fact cyclic.

**Lemma 2.8.** *Given a real vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j-1, j\}$ . Then  $\alpha_1(7) = a_1$ ,  $\alpha_2(7) = a_2$ ,  $\alpha_3(7) = a_3$ ,  $\beta_1(7) = b_1$ ,  $\beta_2(7) = b_2$ ,  $\beta_3(7) = b_3$ .*

*Proof.* By routine computation.  $\square$

The following lemma gives a necessary condition for a matrix to be full-rank.

**Lemma 2.9.** *Given a real vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j-1, j\}$ . Then  $\alpha_1(2) + \beta_1(2) \leq \alpha_2(2) + \beta_2(2)$  implies that  $\alpha_2(6) + \beta_2(6) < \alpha_3(6) + \beta_3(6)$ .*

*Proof.* A routine computation shows that

$$\alpha_2(2) + \beta_2(2) - \alpha_1(2) - \beta_1(2) = \frac{(-a_3 + a_2(1-b_3)) V_{32} V_{21}}{V_{31} V_{73} V_{42} V_{52}},$$

so the sign of  $\alpha_2(2) + \beta_2(2) - \alpha_1(2) - \beta_1(2)$  equals that of  $-a_3 + a_2(1 - b_3)$ . Similarly,

$$\alpha_3(6) + \beta_3(6) - \alpha_2(6) - \beta_2(6) = \frac{V_{46}(-a_3 + a_1(1 - b_3))}{V_{15}V_{36}},$$

so the sign of  $\alpha_3(6) + \beta_3(6) - \alpha_2(6) - \beta_2(6)$  is that of  $-a_3 + a_1(1 - b_3)$ . It remains to note that  $1 - b_3 = V_{42} > 0$  and  $a_1 - a_2 = V_{37} > 0$ .  $\square$

In fact, we can obtain a stronger condition that holds for full-rank matrices.

**Lemma 2.10.** *Given a real vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j - 1, j\}$  and has full nonnegative rank. Then either  $\alpha_1(t) + \beta_1(t) < \alpha_2(t) + \beta_2(t) < \alpha_3(t) + \beta_3(t)$  for every  $t$  or  $\alpha_1(t) + \beta_1(t) > \alpha_2(t) + \beta_2(t) > \alpha_3(t) + \beta_3(t)$  for every  $t$ .*

*Proof.* Assume that  $\alpha_1(t) + \beta_1(t) \leq \alpha_2(t) + \beta_2(t)$ , for some  $t$ . Applying Lemma 2.9 to the vector  $\psi' = (\alpha_1(t+5), \alpha_2(t+5), \alpha_3(t+5), \beta_1(t+5), \beta_2(t+5), \beta_3(t+5))$  and taking into account Lemma 2.8, we obtain that  $\alpha_2(t+4) + \beta_2(t+4) < \alpha_3(t+4) + \beta_3(t+4)$ . Lemma 2.4 then shows that  $\alpha_1(t+4) + \beta_1(t+4) < \alpha_2(t+4) + \beta_2(t+4)$ , and we conclude that  $\alpha_1(t+4k) + \beta_1(t+4k) < \alpha_2(t+4k) + \beta_2(t+4k) < \alpha_3(t+4k) + \beta_3(t+4k)$ , for any positive integer  $k$ .

Now assume  $\alpha_1(t) + \beta_1(t) > \alpha_2(t) + \beta_2(t)$ . By Lemma 2.4, we have  $\alpha_2(t) + \beta_2(t) > \alpha_3(t) + \beta_3(t)$ , and so by Lemma 2.9,  $\alpha_1(t+3) + \beta_1(t+3) > \alpha_2(t+3) + \beta_2(t+3)$ . Finally, we conclude that  $\alpha_1(t+3k) + \beta_1(t+3k) > \alpha_2(t+3k) + \beta_2(t+3k) > \alpha_3(t+3k) + \beta_3(t+3k)$ , for any positive  $k$ .  $\square$

Finally, let us show that a matrix  $\mathcal{V}(\psi)$  can not have full nonnegative rank.

**Lemma 2.11.** *Given a vector  $\psi = (a_1, a_2, a_3, b_1, b_2, b_3)$  for which the matrix  $V = \mathcal{V}(\psi)$  satisfies  $V_{ij} > 0$  if  $i \notin \{j - 1, j\}$ . Then  $\mathcal{V}$  has nonnegative rank less than 7.*

*Proof.* Assume the converse and apply the results of Lemma 2.3 and Lemma 2.10. We can assume without a loss of generality that  $\alpha_1(t) + \beta_1(t) < \alpha_2(t) + \beta_2(t) < \alpha_3(t) + \beta_3(t)$ , for any nonnegative integer  $t$ . Note that  $\alpha_3(0) + \beta_3(0) - \alpha_1(0) - \beta_1(0) = a_3 + b_3 - a_1 - b_1$ , and routine computations also allow us to check that

$$\begin{aligned} \alpha_2(1) + \beta_2(1) - \alpha_1(1) - \beta_1(1) &= \frac{V_{13}(b_1 + (a_1 - 1)b_3)}{V_{63}V_{42}}, \\ \alpha_2(2) + \beta_2(2) - \alpha_1(2) - \beta_1(2) &= \frac{V_{13}V_{21}(a_2(1 - b_3) - a_3)}{V_{73}V_{31}V_{42}V_{52}}. \end{aligned}$$

Noting that also  $1 - b_3 = V_{42} > 0$  and  $V_{37} = a_1 - a_2 > 0$ , we obtain

$$(2.1) \quad b_3(1 - a_1) < b_1, \quad a_3 < a_2(1 - b_3), \quad a_3 + b_3 > a_1 + b_1, \quad b_3 < 1, \quad \text{and} \quad a_1 > a_2.$$

Now let us check that (2.1) is a contradiction. In fact, the first of these inequalities implies  $a_1 + b_1 > a_1 + b_3 - b_3a_1$ , taking into an account the third we obtain  $a_3 + b_3 > a_1 + b_3 - b_3a_1$ . Thus we have  $a_3 > a_1(1 - b_3)$ , which implies  $a_3 > a_2(1 - b_3)$  because of the last two inequalities.  $\square$

Let us now check that 7-by-7 matrices of a more general form have nonnegative rank at most 6 as well. By  $U[r_1, r_2, r_3 | c_1, c_2, c_3]$  we denote the submatrix of  $U$  formed by the rows with indexes  $r_1, r_2, r_3$  and columns with  $c_1, c_2, c_3$ .

**Lemma 2.12.** *Assume that a 7-by-7 matrix  $U$  has classical rank 3 and satisfies  $U_{ij} = 0$  if  $i \in \{j - 1, j\}$  and  $U_{ij} > 0$  otherwise. Then  $U$  has nonnegative rank less than 7.*

*Proof.* Denote by  $U'$  the matrix obtained from  $U$  by multiplying the third column by  $U_{54}/U_{53}$ , the fifth column by  $U_{24}/U_{25}$ , the third row by  $\frac{U_{25}}{U_{24}U_{35}}$ , the fourth row by  $\frac{U_{53}}{U_{43}U_{54}}$ , the  $i'$ th row by  $1/U_{i'4}$  (for  $i'$  from 1, 2, 5, 6, 7). So we have

$$U' = \begin{pmatrix} 0 & 0 & a_3 & 1 & b_3 & U'_{16} & U'_{17} \\ U'_{21} & 0 & 0 & 1 & 1 & U'_{26} & U'_{27} \\ U'_{31} & U'_{32} & 0 & 0 & 1 & U'_{36} & U'_{37} \\ U'_{41} & U'_{42} & 1 & 0 & 0 & U'_{46} & U'_{47} \\ U'_{51} & U'_{52} & 1 & 1 & 0 & 0 & U'_{57} \\ U'_{61} & U'_{62} & a_1 & 1 & b_1 & 0 & 0 \\ 0 & U'_{72} & a_2 & 1 & b_2 & U'_{76} & 0 \end{pmatrix}.$$

Since  $U'$  has classical rank 3, there are certain real constants  $c_1, \dots, c_7$  such that  $U'_{ij} = c_j \det U'[i, j-1, j|3, 4, 5]$ , for any  $i$  and  $j$ . Therefore, we obtain  $U'_{ij} = c_j V_{ij}$  for any  $i$  and  $j$ , where  $V$  is the matrix  $\mathcal{V}(a_1, a_2, a_3, b_1, b_2, b_3)$  from Notation 2.2. Since  $V_{13} = V_{32}$  and  $V_{72} = V_{21}$ , the numbers  $c_1$ ,  $c_2$ , and  $c_3$  are of the same sign. Similarly,  $V_{65} = V_{46}$  and  $V_{76} = V_{57}$ , so that the numbers  $c_5$ ,  $c_6$ , and  $c_7$  are of the same sign as well. Further, since  $V_{24} = V_{25} = V_{43} = 1$ , we obtain  $c_3 = c_4 = c_5 = 1$ , and the numbers  $c_1, \dots, c_7$  are thus all positive. So we can conclude that  $U$  and  $V$  coincide up to multiplying the rows and columns by positive numbers, and the result then follows from Lemma 2.11.  $\square$

Now we can prove the main result of the present section.

**Theorem 2.13.** *A slack matrix of a convex heptagon has nonnegative rank at most 6.*

*Proof.* Proposition 1.1 shows that the slack matrix  $S$  of a convex heptagon has classical rank equal to 3. Therefore,  $S$  satisfies the assumptions of Lemma 2.12 up to renumbering the rows and columns.  $\square$

### 3. MAIN RESULTS

In this section we prove the main results of our paper. Let us start with a corollary of Theorem 2.13 which gives a positive answer for Problem 1.3 in the case  $n = 7$ .

**Theorem 3.1.** *Let  $A$  be a nonnegative 7-by- $n$  matrix with classical rank equal to 3. Then the nonnegative rank of  $A$  does not exceed 6.*

*Proof.* Consider the standard simplex  $\Delta$  consisting of points  $(x_1, \dots, x_7)$  with non-negative coordinates satisfying  $\sum_{i=1}^7 x_i = 1$ . Since  $\Delta$  contains 7 facets, the intersection of  $\Delta$  with the column space of  $A$  is a polygon  $I$  with  $k$  vertices, and  $k \leq 7$ . Form a matrix  $S$  of column coordinate vectors of vertices of  $I$ , then  $A = SB$  with  $B$  nonnegative. If  $k < 7$ , then the result follows directly from that  $A = SB$ , and if  $k = 7$ , then by Theorem 2.13,  $S$  has nonnegative rank less than 7 being a slack matrix for  $I$ .  $\square$

Now we can provide a nontrivial upper bound for the nonnegative rank of matrices with classical rank equal to 3, thus providing a negative solution for Problem 1.3 in the case  $n \geq 7$ .

**Theorem 3.2.** *The nonnegative rank of a rank-three matrix  $A \in \mathbb{R}_+^{m \times n}$  does not exceed  $\left\lceil \frac{6 \min\{m, n\}}{7} \right\rceil$ .*

*Proof.* By Theorem 3.1, any seven rows of  $A$  can be expressed as linear combinations with nonnegative coefficients of certain six nonnegative rows, so the nonnegative rank of  $A$  does not exceed  $\lceil \frac{6m}{7} \rceil$ . The nonnegative rank is invariant under transpositions, so the result follows.  $\square$

Together with the result from [6], where it was noted that a sufficiently irregular convex hexagon has full extension complexity, Theorem 3.2 provides a full answer for Problem 1.3. Namely, the following result is true.

**Theorem 3.3.** *If  $n \geq 7$ , then the nonnegative rank of any rank-three  $m$ -by- $n$  nonnegative matrix is less than  $n$ . For  $k \in \{3, 4, 5, 6\}$ , there are  $k$ -by- $k$  rank-three matrices with nonnegative rank equal to  $k$ .*

Finally, we can prove an upper bound for the extension complexity of convex polygons.

**Theorem 3.4.** *The extension complexity of any convex  $n$ -gon does not exceed  $\lceil \frac{6n}{7} \rceil$ .*

*Proof.* By Proposition 1.1 and Theorem 3.2, the nonnegative rank of a slack matrix does not exceed  $\lceil \frac{6n}{7} \rceil$ , so the result follows from Theorem 1.2.  $\square$

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NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 20 MYASNITSKAYA ULITSA, MOSCOW 101000, RUSSIA

*E-mail address:* yaroslav-shitov@yandex.ru